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AN OSCILLATING SPHERE IN A ROTATING VISCOUS FLUID

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On Oscillating Sphere in a Rotating Viscous Fluid

M. P. Singh
Theoretical Division
NASA-Goddard Space Flight Center
Greenbelt, Maryland

The moderate and large-scale motions of the atmosphere are greatly influenced by the vorticity of the earth's rotation. In the case of an infinite liquid, rotating as a rigid body about an axis, the amount of energy possessed by the liquid is infinite and it is of great interest to know how small disturbances propagate in such a liquid. To understand some of these phenomena, it will be interesting to study the flow of a rotating fluid around elementary bodies.

Here we consider small oscillations of a sphere in an incompressible, viscous fluid rotating with a constant angular velocity. For flow in the absence of rotation, the problem was first considered by Stokes [Lamb, 1932] on the assumption that the Reynolds number $R_s = a|V_s|/\gamma$ was negligibly small (so as to discard convective terms), where V_s is the velocity of the sphere, a is the radius of the sphere and γ the kinematic viscosity. It is the purpose of the present investigation to determine the effect of the coriolis forces on the flow pattern.

Let the sphere oscillate along the axis of the rotating fluid which is taken to be the x-axis. The origin is at the mean position of the center of the sphere. The equations of motion of an incompressible, viscous, unbounded liquid rotating about x-axis with a constant angular velocity $\Omega \underline{i}$ referred to a rotating frame of reference are

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} + 2 \, \underline{\Omega} \times \underline{v} + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}) = - \frac{1}{\rho} \nabla p + \gamma \nabla^2 \underline{v} \tag{1}$$

where ρ is the density, $\underline{v}$ the velocity of the fluid, p is the pressure and r is the radial coordinate:

$$r^2 = y^2 + z^2 \quad .$$

The equation of continuity is

$$\mathrm{div}\, \underline{v} = 0 \quad . \tag{2}$$

As in the classical problem, the convective terms in (1) will henceforth be neglected. This is justified if

$$R_s = a|V_s|/\gamma \ll 1 \quad . \tag{3}$$

The equations (1) and (2) will be reduced to a non-dimensional form by referring the length to a, the radius of the sphere, the velocity to $a\lambda$ ($\lambda/2\pi$ is the frequency of the oscillation), the time to $1/\lambda$ and the pressure to $\rho \, a^2 \, \lambda^2$. Then we have

$$\frac{\partial \underline{v}}{\partial t} = - \nabla p + \delta \, \underline{v} \times \underline{i} + \frac{1}{R_e} \nabla^2 \underline{v} \tag{4}$$

and

$$\text{div } \underline{v} = 0 \tag{5}$$

where

$$R_e \text{(Reynolds number)*} = a^2\lambda/\gamma \tag{6}$$

and

$\delta = 2\,\Omega/\gamma$, the inverse of the so-called Rossby number

$$R_o = \frac{\lambda}{2\Omega} = \frac{1}{\delta} \tag{7}$$

and the centrifugal force $-\delta^2\nabla r^2/8$ has been absorbed in the pressure term.

The boundary conditions on $\underline{v}$ are that $\underline{v} = U_s\underline{i}$ at the sphere $R = 1$

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*It should be noted that we have used $a\lambda$ as velocity in the definition of the Reynolds number. There still is another important dimensionless parameter α/a, where α is the amplitude of the oscillatory motion of the sphere. It is assumed that this quantity is small compared to unity. The velocity of sphere is $V_s\underline{i} = \alpha\,\lambda\, e^{i\lambda t}\,\underline{i}$, or in dimensionless form: $U_s\underline{i} = \alpha/a\; e^{it}\underline{i}$. The Reynolds number for the flow might have also been defined as $R_s = a\,\alpha\,\lambda/\gamma$ $(=R_e\,\alpha/a)$. It is necessary that $R_s \ll 1$, whereas the requirement for R_e is less severe.

(where $R^2 = r^2 + x^2$) and $\underline{v} = 0$ at infinity. (8)

As in non-rotating case, we have here axi-symmetry in that the physical variables are independent of the azimuthal coordinate* Φ. However, the azimuthal component of velocity v_Φ is non-zero here in contrast to the non-rotating case in which v_Φ is zero.

We next make an important assumption in that the Rossby number R_o is large so as to neglect second or higher powers of δ. (9)

From (4) and (5) it follows that $v_x = O(1)$, $v_r = O(1)$ and $v_\Phi = O(\delta)$, hence

$$\nabla^2 p = O(\delta^2) \quad . \tag{10}$$

It follows that the coupled equations (4) can be simplified in this case. In fact, in view of (9) and (10), v_x and v_r satisfy identically the same system of equations and boundary conditions as in non-rotating case. The solution is therefore [Lamb, 1932]

$$p = A \frac{P_1(\cos\theta)}{R^2} \tag{11}$$

and

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*(r, Φ, x) are the cylindrical coordinates.

$$\underline{v} = A\, i\, \nabla \frac{P_1(\cos\theta)}{R^2} + 2B\, f_o\, (hR)\, \nabla\, (R P_1(\cos\theta)) - \qquad (11)$$

$$- B\, f_2\, (hR)\, h^2\, R^5\, \nabla \frac{P_1(\cos\theta)}{R^2}$$

where (R, θ, Φ) are the spherical polar coordinates*, and

$$h^2 = - i\, R_e \quad . \qquad (12)$$

The function $f_n\, (\xi)$, defined as

$$f_n(\xi) = \left(- \frac{1}{\xi} \frac{d}{d\xi}\right)^n \frac{e^{-i\xi}}{\xi} \qquad (13)$$

essentially represents boundary layer effects, while the constants A and B are given by

$$A = - i\, h^2\, f_2\, (h)\, B \quad , \quad B = U_s/2f_o(h) \quad . \qquad (14)$$

Finally, v_Φ remains to be determined, which satisfies

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + h^2\right] v_\Phi = \delta\, R_e v_r \qquad (15)$$

and the boundary conditions

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*$x = R \cos \theta$, $r = R \sin \theta$.

$$v_\Phi = 0 \text{ at } R = 1 \text{ and at infinity.} \tag{16}$$

The solution of (15) and (16) is

$$v_\Phi = -\frac{3}{2}\,\delta\, BR_e\, x\, r\, f_1(hR) + 3\,\delta\, A\,\frac{xr}{R^5} + c\, x\, r\, f_2(hR) \tag{17}$$

where

$$c = 3B\,\delta\, R_e\left[1 + \frac{f_1(h)}{2f_2(h)}\right] . \tag{18}$$

To summarize, the following conclusions are noteworthy:

(i) The azimuthal component of velocity is non-zero in contrast to non-rotating case in which $v_\Phi = 0$.

(ii) The rotating effect generates vorticity in that for large distances, the flow pattern is irrotational in non-rotating case. In rotating case, the azimuthal component of velocity is responsible for vorticity for large R.

(iii) The wave-like terms $f_n(hR)$ (corresponding to a boundary layer) appearing in the solution, effective in a small distance from the surface of the sphere and behaving like $\exp\left[-(1+i)\sqrt{R_e/2}(R-1)\right]$ are essentially similar in nature in rotating and non-rotating cases.

(iv) The drag on the sphere remains unchanged (due to rotation) to the order δ since v_x, v_r and p remain unchanged to that order.

In a forthcoming paper, the above discussion will be extended to a compressible fluid in the presence of a magnetic field.

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Reference

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